SOLVABILITY FOR A CLASS OF NON-LOCAL SEMI-LINEAR DIFFERENTIAL EQUATIONS WITH DELAYS

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ABSTRACT: *In this article, we examine the solvency of mild solutions for a class of non-local semi-linear differential aquations involving finite delays. We prove the global solvability of mild solutions to this problem based on local estimates and fixed point arguments, in which the non-linearity is sub-linear without the smallness condition on initial data as well as on coefficients.*

Keywords: *Global solvability, Non-loca semi-linear equations, Fixed point theory.*

TÓM TẮT: *Trong bài báo này, chúng tôi nghiên cứu tính giải được của một lớp phương trình vi phân không địa phương nửa tuyến tính có trễ hữu hạn. Chúng tôi chứng minh sự tồn tại toàn cục của nghiệm phân tích dựa trên các ước lượng và nguyên lí điểm bất động, ở đó phần phi tuyến là dưới tuyến tính và không cần đến điều kiện dữ kiện ban đầu cũng như các hệ số phải nhỏ.*

Từ khóa: *Tính giải được toàn cục, Phương trình không địa phương nửa tuyến tính, Định lí điểm bất động.*

1. INTRODUCTION

Recently, some authors have already drawn attention to the nonlocal equations. The results on local existence and stability of the solution to the inverse problems were obtained in [1, 3, 4]. It should be noted that, in the mentioned works, no attempt has been made to consider global solvability. This is the main motivation for our study in the present paper.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following problem.

$$\begin{aligned} &\partial_t u = \partial_t (m * \Delta u) + f(t, u_t); \\ &t \geq 0, x \in \Omega \end{aligned} \tag{1.1}$$

with initial condition

$$u(s,x) = \varphi(s,x); s \in [-h,0], x \in \Omega \tag{1.2}$$

for $\varphi \in C([-h,0]; L^2(\Omega))$, $\varphi(0,.) = u_0$, and the Dirichlet boundary condition

$$u(t,x) = 0, t \geq 0; x \in \partial\Omega \tag{1.3}$$

where the kernel $m \in L^1_{loc}(\mathbb{R}^+)$, $f : \mathbb{R}^+ \times C([-h,0]; L^2(\Omega)) \to \mathbb{R}$ is a given function. Here $u_t = u(t+s)$ is the delay term with the interval of delay time $[-h,0]$, and the notation * denotes the Laplace convolution with respect to the time t, i.e.,

$$(\kappa * \nu)(t,x) = \int_0^t \kappa(t-s)\nu(s,x)ds .$$

Equation (1.1) is a class of nonlocal partial differential equations (PDEs), which has been employed to describe anomalous diffusion processes, as remarked in [8]. In the case, function m is differentiable (1.1) becomes a diffusion equation with memory, which describes mass transport in a relaxing medium such as is encountered in polymers near and below their glass transition [2] and it has been proposed in [7] to model the behavior of certain viscoelastic fluids.

In order to deal with (1.1) we use the following standing hypothesis.

(M) The kernel function $m \in L^1_{loc}(\mathbb{R}^+)$, and there exists a non-negative and non-increasing function $k \in L^1_{loc}(\mathbb{R}^+)$ such that $m * k = 1$ on $(0, +\infty)$.

Noting that, the hypothesis (M) was used in a lot of works. This enables us to transform equations of type (1.1) to a Volterra integral equation with completely positive kernel. In this case, one writes $(m, k) \in \mathcal{PC}$. Some typical examples of (m, k) were given in [8], e.g.,

* $m(t) = g_{1-\alpha}(t) := t^{-\alpha} / \Gamma(1-\alpha), \alpha \in (0,1)$ and $k(t) = g_\alpha(t), t > 0$: slow diffusion (fractional order) case.

* $m(t) = \int_0^1 g_\beta(t) d\beta$ and $k(t) = \int_0^\infty \frac{e^{-pt}}{1+p} dp, t > 0$: ultra-slow diffusion (distributed order) case.

* $m(t) = g_{1-\alpha}(t) e^{-\gamma t}, \gamma \geq 0$, and $k(t) = g_\alpha(t) e^{-\gamma t} + \gamma \int_0^t g_\alpha(s) e^{-\gamma s} ds, t > 0$: tempered fractional order case.

The rest of this paper is organized as follows. In the next section, we briefly present some basic notations and give a presentation of mild solutions to (1.1)-(1.3). Section 3 is devoted to showing the results on global solvability in case the nonlinear function f satisfies the sub-linear condition.

2. PRELIMINARIES

To define the resolvent operator and construct the solution formula. We consider the following scalar integral equations

$$s(t) + \mu(m * s)(t) = 1 \tag{2.1}$$

$$r(t) + \mu(m * r)(t) = m(t), t \geq 0. \tag{2.2}$$

The existence and uniqueness of s were analyzed in [6]. Recall that the function m is a completely positive kernel, then $s(\cdot)$ take non-negative values for every $\mu > 0$.

Denote by $s(\cdot, \mu)$ and $r(\cdot, \mu)$ the solutions of (2.1) and (2.2) respectively. As mentioned in [9], the functions $s(\cdot, \mu)$ take non-negative values even in the case $\mu \leq 0$. We collect some additional properties of these functions.

Proposition 2.1. [5,9] Let m is a completely positive function. Then for every $\mu > 0$, $s(\cdot, \mu) \in L^1_{loc}(\mathbb{R}^+)$. In addition, we have:

(1) *The function $s(\cdot, \mu)$ is non-negative and non-increasing. Moreover,*

$$s(t, \mu)\left[1 + \mu \int_0^t m(\theta) d\theta\right] \leq 1, \forall t \geq 0, \tag{2.3}$$

hence if $m \notin L^1(\mathbb{R}^+)$ then $\lim_{t \to \infty} s(t, \mu) = 0$ for every $\mu > 0$.

(2) *The function* $r(\cdot,\mu)$ *is non-negative and one has*

$$s(t,\mu)=1-\mu\int_0^t r(\tau,\mu)d\tau\ t\geq 0, \qquad (2.4)$$

(3) *For each* $t>0$, *the function* $\mu\mapsto s(t,\mu)$ *is non-increasing.*

(4) *Equation* (2.1) *is equivalent to the problem*

$$\frac{ds}{dt}+\mu(m*s)=0, s(0)=1.$$

Remark 2.1. Let

$$w(t)=s(t,\mu)\xi+(s(\cdot,\mu)*g)(t),$$

here

$g\in L^1_{loc}(\mathbb{R}^+)$. If m is a completely positive function, then w is the unique solution of the problem

$$w'+\mu(m*w)'=g(t), w(0)=\xi,\mu>0,t\geq 0. \qquad (2.5)$$

Indeed, take integral to both sides of (2.5), *we have*

$$w+\mu(m*w)=\xi+1*g(t). \qquad (2.6)$$

Taking the Laplace transform on (2.6), *one gets*

$$\hat{w}+\mu\hat{m}\hat{w}=\lambda^{-1}\xi+\lambda^{-1}\hat{g},$$

here $\hat{f}$ *is the Laplace transform of a function* f. *Then*

$$\hat{w}=\lambda^{-1}(1+\mu\hat{m})^{-1}\xi+\lambda^{-1}(1+\mu\hat{m})^{-1}\hat{g}$$
$$=\hat{s}\xi+\hat{s}\hat{g},$$

thanks to the fact that $\hat{s}=\lambda^{-1}(1+\mu\hat{m})^{-1}$. *So taking the inverse Laplace transform on the last relation, we obtain* $w(t)=s(t,\mu)\xi+(s(\cdot,\mu)*g)(t)$.

Suppose that v,w *are two solutions of problem* (2.5) *denote* $\omega=v-w$ *then*

$$\omega'+\mu(m*\omega')=0,\omega(0)=0,$$

this implies $w=0$.

Let $\{\xi_n\}$ be the orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of the operator $-\Delta$ with homogeneous Dirichlet boundary condition, i.e.,

$-\Delta\xi_n=\lambda_n\xi_n$ in Ω, $\xi_n=0$ on $\partial\Omega$, where we can assume that $\{\lambda_n\}$ is non-decreasing with $\lambda_1>0$. Let

$V_\beta=\{v\in L^2(\Omega):\sum_{n=1}^{\infty}\lambda_n^{2\beta}(v,\xi_n)^2<\infty\}$, then V_β is a Hilbert space with the inner product

$$(u,v)_{V_\beta}=\sum_{n=1}^{\infty}\lambda_n^{2\beta}(u,\xi_n)(v,\xi_n).$$

Furthermore, for any $\beta>0$, the embedding $V_\beta\subset L^2(\Omega)$ is compact and $V_{-\beta}$ can be identified with the dual space of V_β. We now define the following operators

$$S(t)x=\sum_{n=1}^{\infty}s(t,\lambda_n)(x,\xi_n)\xi_n,\quad t\geq 0,x\in L^2(\Omega), \qquad (2.7)$$

It is easily seen that $S(t)$ is linear. We show some basic properties of these operators in the following lemma.

Lemma 2.2. *Let* $\{S(t)\}_{t\geq 0}$ *be the families of linear operators defined by* (2.7)

a. For each $x\in L^2(\Omega)$ *and* $T>0$, $S(\cdot)x\in C([0,T];L^2(\Omega))$ *and*

$$\|S(t)x\|_{\mathcal{L}}\leq s(t,\lambda_1)\|x\|, t\in[0,T].$$

b. For each $x \in L^2(\Omega)$,

$(-\Delta)S(\cdot)x \in C((0,T];L^2(\Omega))$ *and*

$$\| S(t)x \|_V \le \frac{\| x \|}{(1*m)(t)}, t \in (0,T]. \qquad (2.8)$$

c. For $g \in C([0,T];\ L^2(\Omega))$, $(-\Delta)^{\frac{1}{2}} S * g \in C([0,T];L^2(\Omega))$ *we have*

$$\| (-\Delta)^{\frac{1}{2}} S * g(t) \|^2 \le \int_0^t k(\tau)d\tau \int_0^t s(t-\tau,\lambda_1) \| g(\tau) \|^2 \, d\tau \qquad (2.9)$$

d. If m *is non-increasing, then* $S(\cdot)v \in C^1((0,T];L^2(\Omega))$ *and it holds that*

$$\| S'(t) \|_{\mathcal{L}} \le t^{-1}, \forall t \in (0,T].$$

Here, the notation $\| \cdot \|$ *stands for the standard norm in* $L^2(\Omega)$ *and* $\| \cdot \|_{\mathcal{L}}$ *represents the operator norm of bounded linear operator acting on* $L^2(\Omega)$.

Proof. The proof of a. and b. can be found in [5].

c. We see that

$$(-\Delta)^{\frac{1}{2}} S * g(t) = \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} \int_0^t s(t-\tau,\lambda_n) g_n(\tau) d\tau e_n, \qquad (2.10)$$

where $g_n(t) = (g(t), e_n)$. Using Hölder inequality, we get

$$\lambda_n \left(\int_0^t s(t-\tau,\lambda_n) g_n(\tau) d\tau \right)^2 \le \lambda_n \int_0^t s(t-\tau,\lambda_n) d\tau \int_0^t s(t-\tau,\lambda_n) | g_n(\tau) |^2 \, d\tau. \qquad (2.11)$$

Recall that

$$s(t,\lambda) + \lambda m * s(\cdot,\lambda)(t) = 1,$$

then $\lambda m * s(\cdot,\lambda)(t) \le 1$. So, we have

$$\begin{aligned} \lambda \int_0^t s(t-\tau,\lambda) d\tau &= \lambda s(\cdot,\lambda) * 1(t) \\ &= \lambda s(\cdot,\lambda) * m * k(t) \\ &\le 1 * k(t). \end{aligned} \qquad (2.12)$$

Combine (2.11), (2.12), ones get

$$\lambda_n \left(\int_0^t s(t-\tau,\lambda_n) g_n(\tau) d\tau \right)^2 \le \int_0^t k(\tau) d\tau \int_0^t s(t-\tau,\lambda_n) | g_n(\tau) |^2 \, d\tau$$

so

$$\begin{aligned} \| (-\Delta)^{\frac{1}{2}} S * g(t) \|^2 &= \sum_{n=1}^{\infty} \lambda_n \left(\int_0^t s(t-\tau,\lambda_n) g_n(\tau) d\tau \right)^2 \\ &\le \int_0^t k(\tau) d\tau \sum_{n=1}^{\infty} \int_0^t s(t-\tau,\lambda_1) | g_n(\tau) |^2 \, d\tau \\ &= \int_0^t k(\tau) d\tau \int_0^t s(t-\tau,\lambda_1) \| g(\tau) \|^2 \, d\tau, \end{aligned}$$

which implies (2.9). In order to show $(-\Delta)^{\frac{1}{2}} S * g \in C([0,T];L^2(\Omega))$, it suffices to check that series (2.10) is uniformly convergent on $[0,T]$. Since g is continuous, the series $\sum_{n=1}^{\infty} |g_n(\tau)|^2$ is uniformly convergent on $[0;\ T]$. That means, for any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $\sum_{n=N}^{N+p} |g_n(\tau)|^2 \le \epsilon$ for all $N > N_\epsilon, p \in \mathbb{N}$ and $\tau \in [0;T]$. It follows that

$$\begin{aligned} &\sum_{n=N}^{N+p} \lambda_n \int_0^t s(t-\tau,\lambda_n) g_n(\tau) d\tau \\ &\le \int_0^t k(\tau) d\tau \sum_{n=N}^{N+p} \int_0^t s(t-\tau,\lambda_1) | g_n(\tau) |^2 \, d\tau \\ &= \int_0^t k(\tau) d\tau \int_0^t s(t-\tau,\lambda_1) \sum_{n=N}^{N+p} |g_n(\tau)|^2 \, d\tau \end{aligned}$$

$$\leq \epsilon\int_0^t k(\tau)d\tau\int_0^t s(t-\tau,\lambda_1)d\tau$$
$$\leq \epsilon\int_0^T k(\tau)d\tau\int_0^T s(t-\tau,\lambda_1)d\tau,$$

for all $t\in[0;T]$, which guarantees the uniform convergence of (2.10) on [0; T].

d. Let $r(\cdot,\lambda)$ be the solution of (2.2) with $\mu=\lambda$. Then, due to the assumption that m is non-increasing, we have

$$r(\cdot,\lambda)+\lambda m(t)\int_0^t r(\tau,\lambda)d\tau \leq m(t).$$

In addition, it follows from (2.4) that

$$\int_0^t r(\tau,\lambda)d\tau = \lambda^{-1}(1-s(t,\lambda))$$
$$\geq \lambda^{-1}\left(1-\frac{1}{1+\lambda(1*m)(t)}\right)$$
$$=\frac{(1*m)(t)}{1+\lambda(1*m)(t)}.$$

Hence

$$r(t,\lambda)\leq m(t)\left(1-\lambda\int_0^t r(\tau,\lambda)d\tau\right)$$
$$\leq m(t)\left(1-\lambda\frac{(1*m)(t)}{1+\lambda(1*m)(t)}\right) \quad (2.13)$$
$$=\frac{m(t)}{1+\lambda(1*m)(t)}$$

Considering the series

$$\sum_{n=1}^{\infty} s'(t,\lambda_n)\psi_n\xi_n, t>0, \xi_n = (\psi,\xi_n), \psi\in L^2(\Omega), \quad (2.14)$$

we see that

$$|s'(t,\lambda_n)| = \lambda_n r(t,\lambda_n)$$
$$\leq \frac{\lambda_n m(t)}{1+\lambda_n(1*m)(t)}$$
$$\leq \frac{\lambda_n m(t)}{1+\lambda_n t m(t)}$$
$$\leq \frac{1}{t},$$

thanks to (2.13) and the fact that $(1*m)(t)\geq tm(t)$ for $t>0$. This ensures the uniform convergence of series (2.14) on $[\epsilon;T]$ and it holds that

$$S'(t)\psi=\sum_{n=1}^{\infty} s'(t,\lambda_n)\psi_n\xi_n, \| S'(t)\psi \|$$
$$\leq t^{-1}\|\psi\|, t>0, \psi\in L^2(\Omega).$$

It follows from (2.8) that, for any $t>0$, $S(t): L^2(\Omega)\to L^2(\Omega)$ is compact. Furthermore, we will prove that the Cauchy operator

$$Q: C([0,T];L^2(\Omega))\to C([0,T];L^2(\Omega))$$

define by

$$Q(g)(t)=(S*g)(t) \quad (2.15)$$

is compact in the next lemma.

Lemma 2.3. *Let the assumption* (M) *hold. If the function* m *is non-increasing, then the operator* Q *defined by* (2.15*) is compact.*

Proof. Let D be a bounded set in $C([0,T];L^2(\Omega))$. We first show that $(-\Delta)^{\frac{1}{2}}Q(D)(t)$ is a bounded set in $L^2(\Omega)$ for each $t\geq 0$. By using Lemma 2.2, we have

$$\|(-\Delta)^{\frac{1}{2}}Q(g)(t)\|^2$$
$$\leq \int_0^t k(\tau)d\tau\left(\int_0^t s(t-\tau,\lambda_1)\|g(\tau)\|^2 d\tau\right),$$
$$\forall t\in[0;T],$$

which ensures the boundness of $(-\Delta)^{\frac{1}{2}}Q(D)(t)$ in $L^2(\Omega)$ for all $t\geq 0$.

Since the embedding $D((-\Delta)^{\frac{1}{2}}) \to L^2(\Omega)$ is compact, we obtain the compactness of $Q(D)(t)$ for each $t \geq 0$.

Now we prove that $Q(D)$ is equicontinuous. Let $g \in D, t \in (0,T]$ and $\theta \in (0, T-t]$, then

$$\begin{aligned}
&\| Q(g)(t+\theta) - Q(g)(t) \| \\
&\leq \int_0^t \| [S(t+\theta-\tau) - S(t-\tau)]g(\tau) \| d\tau \\
&\quad + \int_t^{t+\theta} \| S(t+\theta-\tau)g(\tau) \| d\tau \\
&= I_1(t) + I_2(t).
\end{aligned}$$

It is easy to see that $I_2(t) \to 0$ as $\theta \to 0$ uniformly in $g \in D$. Regarding to $I_1(t)$, we observe that

$$\begin{aligned}
&\| [S(t+\theta-\tau) - S(t-\tau)]g(\tau) \| \\
&= \| \int_0^1 \theta S'(t-\tau+\nu\theta) g(\tau) d\nu \| \\
&\leq \theta \int_0^1 \| S'(t-\tau+\nu\theta) \| g(\tau) d\nu \\
&\leq \theta \int_0^1 \frac{\| g(\tau) \| d\nu}{t-\tau+\nu\theta},
\end{aligned}$$

thanks to the mean value formula and Lemma 2.2(d). So

$$\begin{aligned}
&\| [S(t+\theta-\tau) - S(t-\tau)]g(\tau) \| \\
&\leq \| g \|_\infty \ln\left(1 + \frac{\theta}{t-\tau}\right) \qquad (2.16) \\
&\leq \| g \|_\infty \frac{\theta^\beta}{\beta (t-\tau)^\beta}, \beta \in (0,1),
\end{aligned}$$

here $\| g \|_\infty = \sup_{t \in [0,T]} \| g(t) \|$, and using the inequality $\ln(1+r) \leq \frac{r^\beta}{\beta}$ for any $r > 0, \beta \in (0,1)$. Employing (2.16), we obtain

$$\begin{aligned}
I_1(t) &\leq \frac{\| g \|_\infty \theta^\beta}{\beta} \int_0^t \frac{d\tau}{(t-\tau)^\beta} \\
&\leq \frac{\| g \|_\infty \theta^\beta}{\beta(1-\beta)} T^{1-\beta} \to 0 \text{ as } \theta \to 0 \text{ uniformly in } g \in D.
\end{aligned}$$

Finally, for $\theta \in (0,T]$, we have

$$\begin{aligned}
&\| Q(g)(\theta) - Q(g)(0) \| \\
&\leq \int_0^\theta \| S(\theta-\tau) g(\tau) \| d\tau \\
&\leq \theta \| g \|_\infty \to 0 \text{ as } \theta \to 0,
\end{aligned}$$

uniformly in $g \in D$. Therefore, $Q(D)$ is equicontinuous. We have the conclusion due to the Arzel`a-Ascoli theorem.

Concerning the nonlinearity function f, we can refer to f as a map from $\mathbb{R}^+ \times C([-h,0]; L^2(\Omega))$ into $L^2(\Omega)$. Based on the operators $S(t)$, we introduce the following definition of solutions to (1.1) –(1.3) as follows

Definition 2.1. *A function $u \in C([-h,T]; L^2(\Omega))$ is said to be a mild solution to* (1.1) –(1.3) *on* $[-h,T]$ *iff* $u(t) = \varphi(t)$ *for* $t \in [-h,0]$ *and*

$$u(t) = S(t)\varphi(0) + \int_0^t S(t-\theta) f(\theta, u_\theta) d\theta$$

for $t \in [0,T]$.

3. EXISTENCE RESULTS

Let $\mathcal{C}_h = C([-h,0]; L^2(\Omega))$. Then $\mathcal{C}_h$ is a Banach space endowed with the norm

$$\| x \|_{\mathcal{C}_h} = \sup_{s \in [-h,0]} \| x(s) \|.$$

For $u \in C([0,T]; L^2(\Omega))$ and $\varphi \in \mathcal{C}_h$, we denote the function $u[\varphi]$ by

$$u[\varphi](t) = \begin{cases} u(t) & \text{if } t > 0, \\ \varphi(t) & \text{if } -h \leq t \leq 0. \end{cases}$$

Let

$$\mathcal{C}_\varphi = \{u \in C([0,T]; L^2(\Omega)) : u(0) = \varphi(0)\},$$

$$\Sigma(u)(t) = S(t)\varphi(0) + \int_0^t S(t-\theta) f(\theta, u[\varphi]_\theta) d\theta,$$

$u \in \mathcal{C}_\varphi, t \geq 0.$

Then u is a fixed point of Σ iff $u[\varphi]$ is a mild solution of (1.1) – (1.3). So we refer to Σ as the solution operator. In the sequel, we use $\|\cdot\|_\infty$ for the supremum norm in the space $C(J; L^2(\Omega))$, where $J \subset \mathbb{R}$ is a compact interval.

In the following theorem, we show a result on global existence, without the smallness condition on initial data as well as on coefficients.

Theorem 3.1. *Let f be continuous and satisfy the sub-linear condition*

$$\| f(t,w) \| \leq \alpha(t) + \kappa \| w \|_{\mathcal{C}_h}, \quad \forall t \geq 0, w \in \mathcal{C}_h, \tag{3.1}$$

where $\alpha \in L^1_{loc}(\mathbb{R}^+)$ is a non-negative function, κ is non-negative numbers. Then the problem (1.1)- (1.3) *has at least one mild solution.*

Proof. Let us divide the proof into 2 steps.

First, We look for a convex set $\mathcal{M} \subset \mathcal{C}_\varphi$ such that $\Sigma(\mathcal{M}) \subset \mathcal{M}$. By the formulation of Σ, we have

$$\begin{aligned}\|\Sigma(u)(t)\| &\leq s(t,\lambda_1)\|\varphi(0)\| \\ &+\int_0^t s(t-\theta,\lambda_1)\left(\alpha(\theta) + \kappa \|u[\varphi]_\theta\|_{\mathcal{C}_h}\right) d\theta \\ &\leq s(t,\lambda_1)\|\varphi(0)\| + \int_0^t s(t-\theta,\lambda_1)\alpha(\theta) d\theta \\ &+\int_0^t s(t-\theta,\lambda_1)\left(\kappa \|\varphi\|_{\mathcal{C}_h} + \kappa \sup_{[0,\theta]} \|u(\xi)\|\right) d\theta,\end{aligned}$$

$t \in [0,T]$,

thanks to the fact that

$$\begin{aligned}\| v[\varphi]_s \|_{\mathcal{C}_h} &= \sup_{\theta\in[-h,0]} \| v[\varphi](s+\theta) \| \\ &\leq \|\varphi\|_{\mathcal{C}_h} + \sup_{\rho\in[0,s]} \|v(\rho)\|.\end{aligned}$$

Then, in view of Proposition 2.1(1), we get

$$\begin{aligned}\|\Sigma(u)(t)\| &\leq s(t,\lambda_1)\|\varphi(0)\| + (s(\cdot,\lambda_1) * \alpha)(t) \\ &+ \kappa \|\varphi\|_{\mathcal{C}_h} \int_0^t s(t-\theta,\lambda_1) d\theta \\ &+ \kappa \int_0^t s(t-\theta,\lambda_1) \sup_{\xi\in[0,\theta]} \|u(\xi)\| d\theta \\ &\leq \left(1 + \kappa \int_0^t s(\theta,\lambda_1) d\theta\right) \|\varphi\|_{\mathcal{C}_h} \\ &+ \sup_{t\in[0,T]} (s(\cdot,\lambda_1) * \alpha)(t) \\ &+ \kappa \int_0^t s(t-\theta,\lambda_1) \sup_{\xi\in[0,\theta]} \|u(\xi)\| d\theta,\end{aligned}$$

$t \in [0,T]$.

Let

$$M_0 = \left(1 + \kappa \sup_{t\in[0,T]} (1 * s(\cdot,\lambda_1))(t)\right) \|\varphi\|_{\mathcal{C}_h} + \sup_{t\in[0,T]} (s(\cdot,\lambda_1) * \alpha)(t).$$

Since the function $\theta \mapsto \sup_{[0,\theta]} \|u(\xi)\|$ is non-decreasing, the last integral is non-decreasing in t, thanks to the positivity of $s(\cdot,\lambda_1)$. Thus we get

$$\sup_{\xi\in[0,t]} \|\Sigma(u)(\xi)\| \leq M_0 + \kappa \int_0^t s(t-\theta,\lambda_1) \sup_{\xi\in[0,\theta]} \|u(\xi)\| d\theta. \tag{3.2}$$

Let $w \in C([0,T]; \mathbb{R}^+)$ be the unique solution of the integral equation

$$w(t) = M_0 + \kappa \int_0^t s(t-\theta,\lambda_1) w(\theta) d\theta, t \in [0,T].$$

We define the set

$\mathcal{M} = \{u \in \mathcal{C}_{\varphi} : \sup_{\xi\in[0,t]} \| u(\xi) \| \le w(t), \forall t \in [0,T]\}$.

Clearly, $\mathcal{M}$ is a closed convex set. Then inequality (3.2) ensures that $\Sigma(\mathcal{M}) \subset \mathcal{M}$.

Second, to apply Schauder fixed point theorem to the solution operator Σ. We construct a compact and convex set $\mathcal{M}^*$, such that $\Sigma(\mathcal{M}^*) \subset \mathcal{M}^*$. Recall that the map $Q_t : C([0,T]; L^2(\Omega)) \to L^2(\Omega)$ defined by $Q_t(g) = (S * g)(t)$ is compact for each $t \in [0;T]$, we see that

$$\Sigma(\mathcal{M})(t) = S(t)\varphi(0) + Q_t(N_f(\mathcal{M})),$$

is a relatively compact set in $L^2(\Omega)$, here $N_f(\mathcal{M}) = \{f(\cdot, y(\cdot - \theta)) : y \in \mathcal{M}\}$, which is bounded. Due to the compactness of operator Q started in Lemma 2.3 we get that $\Sigma(\mathcal{M})$ is relatively compact.

Let $\mathcal{M}^* = \overline{\mathrm{co}}\Sigma(\mathcal{M})$, then $\mathcal{M}^*$ is a compact and convex set. In addition, observing that $\mathcal{M}^* \subset \mathcal{M}$, one gets $\Sigma(\mathcal{M}^*) \subset \Sigma(\mathcal{M}) \subset \mathcal{M}^*$. Applying the Schauder theorem for $\Sigma : \mathcal{M}^* \to \mathcal{M}^*$, we have the desired conclusion on solvability.

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